

**Hale School**

**MATHEMATICS**

**SPECIALIST**

**3CD**

**Semester Two Examination 2010**

**MARKING KEY and SOLUTIONS**

**Section One**

**Calculator-Free**

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**Question 1 [6 marks]**

Find the anti-derivative :

(a) 

[2 marks]

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| **Solution** |
| = x .  + c =  + c |
| **Specific Behaviours** |
| ✓ Anti-derivative of exponential is itself  ✓ Recognises the derivative factor to divide by 4x |

(b) 

[2 marks]

|  |
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| **Solution** |
| =  =  = |
| **Specific Behaviours** |
| ✓ Integrates power of sine function  ✓ Recognises the derivative factor to divide by cos x |

(c) 

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| **Solution** |
| =  = |
| **Specific Behaviours** |
| ✓ Integrates the power of ln x  ✓ Recognises the derivate factor 1/x |

[2 marks]

**Note : If there is NO use of integration constants in Q1, then ONE mark is to be deducted from Q1c.**

**Question 2 [9 marks]**

Given that z = 2eix and w = 2e–ix :

(a) express iz in complex exponential form.

[2 marks]

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| **Solution** |
| iz =  = |
| **Specific Behaviours** |
| ✓ Converts i into exponential form  ✓ adds indices to give argument x + π/2 and real part 2 |

(b) express  in complex exponential form.

[2 marks]

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| **Solution** |
| =  = |
| **Specific Behaviours** |
| ✓ Expresses cis 5x in exponential form  ✓ subtracts indicies to give argument 4x and real part 0.5 |

(c) simplify z2 + w2

[2 marks]

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| **Solution** |
| z2 + w2 = 4e2ix  + 4e-2ix  = 4(e2ix  + e-2ix)  = 4 . 2 cos 2x  = 8 cos 2x |
| **Specific Behaviours** |
| ✓ Multiply indices by 2  ✓ Express as twice the real part cos 2x |

(d) solve for x given that z3 + 8 = 0

[3 marks]

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| **Solution** |
| z3 = -8 = 8 cis π  ∴ z =  k = 0, 1, 2  ∴ z = , ,  ∴ x = π/3, π, - π/3 |
| **Specific Behaviours** |
| ✓ Expression for cube roots using De Moivre’s Theorem  ✓ Give roots in cis form  ✓ Solve for x |

**Question 3 [4 marks]**

Points A and B have respective position vectors given by :

**a** = 2**i** + **j**  - **k**

**b** = x**i** + **j**  + **k**

Determine the value of x given that vectors **a**  and **b** are at an angle of 60o.

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| **Solution** |
| **a** **.** **b** = 2x + 1 – 1 =  ∴ 2x =  so x > 0 (square root gives a positive)  ∴ (4x)2 = 6x2 + 12  10x2 = 12  x =  But x > 0 ∴ x = |
| **Specific Behaviours** |
| ✓ expresses dot product using components AND using magnitudes  ✓ uses cos 600 = 0.5 to form an equation in x2  ✓ squares both sides to eliminate the square root  ✓ solves to find 2 values for x but only accepts the POSITIVE solution |

**Question 4 [4 marks]**

Give the following transformation matrices, describe their effect on some object in the co-ordinate plane :

(a) 

[1 mark]

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| **Solution** |
| Horizontal dilation about x = 0 with factor 2 |
| **Specific Behaviours** |
| ✓ Uses term dilation about x = 0 and mentions factor 2 |

(b) 

[1 mark]

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| **Solution** |
| Reflection about y = x |
| **Specific Behaviours** |
| ✓ Uses term reflection and states the position of the mirror y = x |

**Question 4 [4 marks]**

(c) 

[2 marks]

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| **Solution** |
| Vertical shear with factor 2 AND THEN an anti-clockwise rotation of 90o about origin |
| **Specific Behaviours** |
| ✓ States 2 transformations and gives the correct order (shear then rotate)  ✓ Describes each transformation correctly. |

**Question 5 [6 marks]**

Prove using the method of mathematical induction that, for all values of the positive integer n, 7n + 2 is always divisible by 3.

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| **Solution** |
| For n = 1, 71 + 2 = 9 which is divisible by 3.  Hence true for n = 1.  Assume true for n = k i.e. 7k + 2 = 3m where m is some integer  Consider n = k + 1 : 7k+1 + 2 = 7.7k + 2  = 7(3m – 2) + 2  = 21m – 12  = 3(7m – 4) which is divisible by 3  Hence true for n = k + 1  Hence true ∀ n. |
| **Specific Behaviours** |
| ✓ Show true for n = 1  ✓ Assumes true for n = k  ✓ Expresses divisibility by 3 using some multiple of 3  ✓ Expresses 7k+1 in terms of previous result  ✓ Shows that result for n = k+1 has a factor of 3  ✓ Concludes that true for all values n |

**Question 6 [7 marks]**

A particle’s position is given by x(t) cm after t seconds and moves according to the differential equation :



It is known that x(0) = k cm, and its velocity v(0) =  cm s-1 where k is some positive constant. Write an expression (in terms of the constant k) for :

(a) the displacement x(t).

[5 marks]

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| **Solution** |
| Motion is SHM from the differential equation with T = 2 seconds  Hence suggest x(t) = A cos(πt + α)  Given x(0) = k then k = A cos α …… (1)  v(t) = - πA sin(nt + α)  Given v(0) = - √3πk then -√3πk = -πA sin α ….. (2)  From (1) cos α = k/A  From (2) sin α = -√3k/A ∴ cos2α + sin2α = 1  ∴  i.e. A = 2k (k > 0)  ∴ cos α = 0.5 i.e. α = π/3  Hence x(t) = 2k cos(πt + π/3) |
| **Specific Behaviours** |
| ✓ States motion as SHM and determines the period T  ✓ Writes x(t) of form A cos(nt + α) i.e. a trigonometric function with phase shift  ✓ Obtains relationships between k and A using x(0) and v(0)  ✓ Solves for A and α.  ✓ Concludes with expression for x(t) in terms of k |

(b) the distance travelled in the first 4 seconds.

[2 marks]

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| **Solution** |
| Since the period T = 2 sec, then over 4 seconds, the particle does 2 complete cycles. Hence Distance = 2(4A) = 16k |
| **Specific Behaviours** |
| ✓ Recognises 2 cycles of motion  ✓ Expresses distance in terms of k |

**Question 7 [4 marks]**

It is known that  is a solution to the equation zn = i. Determine the set of possible values for the positive integer n.

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| **Solution** |
| zn = cis π/2  ∴ z =  k = 0, 1, 2, . . . , n-1  ∴  for some integer value of k and n  ∴ 3n = 2 + 8k  ∴ 3n = 2, 10, **18**, 26, 34, **42**, . . .  ∴ n = 6, 14, 22, . . .  Hence n can be any integer that is 2 less than a multiple of 8 |
| **Specific Behaviours** |
| ✓ Expression for the n-th roots using De Moivre’s Theorem  ✓ Express cis(3π/4) as one of the solutions  ✓ Develops equation to determine n  ✓ Obtains the entire set of values for n |